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Primitive Solvable Groups

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Suppose G is a finite group, $\mathbf{k}$ an algebraically closed field, and V a $\mathbf{k}[G]$-module. We say V is a *primitive* G-module if (1) V is irreducible and (2) V is not induced from a submodule of a proper subgroup of G. That is, G has no system of imprimitivity on V [5(50.1)]. Following Brauer [4] we say that V is *quasi primitive* if (1) V is irreducible and (2) $V|_N$ is homogeneous for every normal subgroup $N \triangle G$. That is, $V|_N$ is a multiple of a single irreducible G-module. It is clear that a primitive module will be quasi-primitive. The converse situation is not so obvious.

If $\mathbf{k}$ is the complex field, G is the alternating group on 5 points, and V is the irreducible module of degree 5 then V is induced from the subgroup A^4. But since G and 1 are the only normal subgroups V is quasiprimitive. That is, V is quasiprimitive but not primitive. However, such examples are restricted to nonsolvable groups. In fact, we have the following.

THEOREM. *In a solvable group primitivity and quasiprimitivity are equivalent.*

Assume V is an irreducible G-module for the solvable group G. If V is quasiprimitive then it is primitive. If V is not quasiprimitive then there is a normal subgroup $N \triangle G$ so that $V|_{N_1} \simeq U_1 \dotplus \cdots \dotplus U_t$ is a sum of more than one homogeneous component U_i. Let G_1 be the stabilizer in G of U_1. Then U_1 is an irreducible G_1-module, say V_1, and $V_1|^G \simeq V$. We may repeat this process for V_1 and G_1. Eventually we will arrive at a quasi-primitive module V_s of a subgroup G_s of G. We may call V_s a *stabilizer limit* for V. Note that $V_s|^G \simeq V$. As an immediate corollary we then obtain the following.

COROLLARY. *A stabilizer limit U for V is a primitive module which induces V.*

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1. Terminology

In this paper modular as well as projective representations are mentioned. The word "ordinary" is therefore used to mean "nonmodular" (i.e., the field characteristic does not divide the group order). The word "linear" is used to mean "nonprojective" (i.e., projective with trivial factor set). A one dimensional module will be called one dimensional rather than "linear." All other notation and terminology is fairly standard.

The next section contains some fairly obvious but nice splitting properties of representations of groups. In the final section the main theorem is proved by an induction upon dimension.

2. Preliminary Results

Let $\mathbf{k}$ be an algebraically closed field. Let G be a group with normal subgroup H. Assume V is an irreducible $\mathbf{k}[G]$-module for which $V|_H$ is irreducible. Let U, W be $\mathbf{k}[G]$-modules which are trivial on H. All tensor products are over $\mathbf{k}$ unless otherwise noted.

Theorem 2.1. *If A, B are $\mathbf{k}[G]$-modules then $\mathrm{Hom}_{\mathbf{k}[H]}(A, B)$ is a $\mathbf{k}[G]$-module with action $x \cdot \phi(v) = x(\phi(x^{-1}v))$ containing H in the kernel.*

First, the module action is obvious. Let $x, y \in G$. Then

$$\begin{aligned} x \cdot (y \cdot \phi)(v) &= x(y \cdot \phi)(x^{-1}v) \\ &= x(y\phi)(y^{-1}x^{-1}v) \\ &= (xy)\,\phi((xy)^{-1}v) \\ &= (xy) \cdot \phi(v). \end{aligned}$$

So $\mathrm{Hom}_{\mathbf{k}[H]}(A, B)$ is a $\mathbf{k}[G]$-module.

Second, for $x \in H$ we have

$$x \cdot \phi(v) = x\phi(x^{-1}v) = (xx^{-1})\,\phi(v) = \phi(v).$$

So H is in the kernel.

Theorem 2.2. *Suppose $L \geqslant H$ is a normal subgroup of G. Then as $\mathbf{k}[G]$-modules*

$$\mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W) \simeq \mathrm{Hom}_{\mathbf{k}[L]}(U, W).$$

If, in addition, L is in the kernel of U and W then

$$\mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W) \simeq \mathrm{Hom}_{\mathbf{k}}(U, W).$$

Once we have the first isomorphism, the second is obvious since the two modules are the same.

By Theorem 2.1 all Hom's are $\mathbf{k}[G]$-modules. Suppose $\phi \in \mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W)$. We show that $\phi = 1 \otimes \phi^*$ for some $\phi^* \in \mathrm{Hom}_{\mathbf{k}[L]}(U, W)$.

Let T be the representation of H on V. Now $V|_H$ is irreducible so that

$$\langle T(x) \mid x \in H \rangle = \mathrm{Hom}_{\mathbf{k}}(V, V)$$

the $T(x)$'s generate $\mathrm{Hom}_{\mathbf{k}}(V, V)$ as a $\mathbf{k}$-algebra. That is, if $\triangle \in \mathrm{Hom}_{\mathbf{k}}(V, V)$ then

$$\triangle = \sum_{x \in H} \alpha_x T(x)$$

for some choice of coefficients $\alpha_x \in \mathbf{k}$.

Let $v_1, \ldots, v_n$; $u_1, \ldots, u_m$; $w_1, \ldots, w_s$ be respectively $\mathbf{k}$-bases for V, U, and W. Then $\{v_i \otimes u_j\}$, $\{v_i \otimes w_j\}$ are, respectively, $\mathbf{k}$-bases for $V \otimes U$ and $V \otimes W$. Let ϕ_{ij} take v_i to v_j and v_k to 0 for $k \neq i$. Let ψ_{ij} take u_i to w_j and u_k to 0 for $k \neq i$. Then we easily extend these to get $\phi_{ij} \in \mathrm{Hom}_{\mathbf{k}}(V, V)$ and $\psi_{ij} \in \mathrm{Hom}_{\mathbf{k}}(U, W)$. Further $\{\phi_{ij} \otimes \psi_{ab}\}$ is a $\mathbf{k}$-basis for $\mathrm{Hom}_{\mathbf{k}}(V \otimes U, V \otimes W)$. Recall that $\phi \in \mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W)$. We may now write

$$\phi = \sum \alpha_{ab}^{ij} \phi_{ij} \otimes \psi_{ab}$$

for some $\alpha_{ab}^{ij} \in \mathbf{k}$. For each $x \in H$, ϕ commutes with x. Now

$$\phi_{ij} = \sum_{x \in H} \alpha_x T(x).$$

Now x acts upon $V \otimes U$ and $V \otimes W$ as $T(x) \otimes 1$. So ϕ commutes with $\phi_{ij} \otimes 1 = \sum \alpha_x T(x) \otimes 1$ for all choices of i, j. Thus

$$\phi(\phi_{cd} \otimes 1) = (\phi_{cd} \otimes 1)\phi.$$

Or

$$(\phi_{cd} \otimes 1)\phi = \sum_{i,j,a,b} \alpha_{ab}^{ij}(\phi_{cd}\phi_{ij}) \otimes \psi_{ab}$$

$$= \sum_{a,b,j} \alpha_{ab}^{dj} \phi_{cj} \otimes \psi_{ab}$$

and

$$\phi(\phi_{cd} \otimes 1) = \sum_{i,j,a,b} \alpha_{ab}^{ij}(\phi_{ij}\sigma_{cd}) \otimes \psi_{ab}$$

$$= \sum_{i,a,b} \alpha_{ac}^{ic} \phi_{id} \otimes \psi_{ab} .$$

Thus $\alpha_{ab}^{ij} = 0$ unless $i = j$ and in that case $\alpha_{ab}^{dd} = \alpha_{ab}^{cc}$. So

$$\begin{aligned}\phi &= \sum_{a,b} \alpha_{ab} 1 \otimes \psi_{ab} \\ &= 1 \otimes \sum_{a,b} \alpha_{ab}\psi_{ab} \\ &= 1 \otimes \phi^*,\end{aligned}$$

where we set $\alpha_{ab}^{cc} = \alpha_{ab}$.

Now $\phi \in \mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W)$. Let $v \in V$, $u \in U$, $x \in L$. Then

$$\begin{aligned}\phi(v \otimes u) &= x \cdot \phi(v \otimes u) = x\phi((x^{-1}v) \otimes (x^{-1}u)) \\ &= x((x^{-1}v) \otimes \phi^*(x^{-1}u)) = v \otimes (x \cdot \phi^*)(u) \\ &= v \otimes \phi^*(u).\end{aligned}$$

So $x \cdot \phi^* = \phi^*$ and $\phi^* \in \mathrm{Hom}_{\mathbf{k}[L]}(U, W)$.

The map $\phi^* \to 1 \otimes \phi^*$ is clearly a $\mathbf{k}[G]$-isomorphism of $\mathrm{Hom}_{\mathbf{k}[L]}(U, W)$ into $\mathrm{Hom}_{\mathbf{k}[L]}(V \otimes U, V \otimes W)$. Our argument above shows it is onto proving Theorem 2.2.

THEOREM 2.3. *Assume U and W are completely reducible $\mathbf{k}[G]$-modules.*

(1) *$V \otimes U$ is irreducible if and only if U is also.*

(2) *If U is irreducible then $V \otimes U \simeq V \otimes W$ if and only if $U \simeq W$.*

Now $\mathrm{Hom}_{\mathbf{k}[G]}(V \otimes U, V \otimes W) \simeq \mathrm{Hom}_{\mathbf{k}[G]}(U, W)$ by Theorem 2.2. Both these modules have the same dimension.

In (1) we take $W = U$. By [5(43.18)] for a completely reducible $\mathbf{k}[G]$-module A, A is irreducible if and only if $\mathrm{Hom}_{\mathbf{k}[G]}(A, A)$ has $\mathbf{k}$-dimension 1. Thus (1) follows.

In (2), $V \otimes U$ is irreducible by (1). So obviously $V \otimes W$ must be irreducible and W must also be irreducible by (1). Now $\mathrm{Hom}_{\mathbf{k}[G]}(U, W)$ has dimension 1 if and only if $U \simeq W$.

THEOREM 2.4. *Assume G has normal subgroups $H \geqslant L$. Suppose $M = A \times L$, $G = AH$ and $A \cap H = 1$. Assume X and Z are, respectively, $\mathbf{k}[A]$- and $\mathbf{k}[L]$-modules. Suppose $(X \otimes Z)|^G \simeq V \otimes U$ is induced from the $\mathbf{k}[AL]$-module $X \otimes Z$, where $V|_H$ is irreducible, $U|_H$ is trivial, and V and U are $\mathbf{k}[G]$-modules. If $V|_{AL} \simeq W \otimes Z$, where W is a $\mathbf{k}[A]$-module then*

$$\mathrm{Hom}_{\mathbf{k}}(W, X) \simeq U$$

is a $\mathbf{k}[A]$-isomorphism.

Let J be the trivial one dimensional $G/H \simeq A$-module. Then $U \simeq \mathrm{Hom}_{\mathbf{k}}(J, U)$ is an A-isomorphism. By Theorem 2.2

$$\mathrm{Hom}_{\mathbf{k}}(J, U) \simeq \mathrm{Hom}_{\mathbf{k}[H]}(V, V \otimes U)$$

is a $\mathbf{k}[G]$-isomorphism with H in the kernel. That is, it is a $k[A]$-isomorphism. Now

$$\mathrm{Hom}_{\mathbf{k}[H]}(V, V \otimes U) \simeq \mathrm{Hom}_{\mathbf{k}[H]}(V, (X \otimes Z)|^G).$$

We show that the $\mathbf{k}$-map (Frobenius Reciprocity Isomorphism) is a $\mathbf{k}[AL]$-isomorphism of $\mathrm{Hom}_{\mathbf{k}[H]}(V, (X \otimes Z)|^G)$ onto $\mathrm{Hom}_{\mathbf{k}[L]}(V|_{AL}, X \otimes Z)$. For $\phi \in \mathrm{Hom}_{\mathbf{k}[H]}(V, (X \otimes Z)|^G)$ and $v \in V$,

$$\phi(v) = \sum x_i \otimes_{\mathbf{R}} v_i, \qquad \text{where} \quad \mathbf{R} = \mathbf{k}[AL],$$

$\{x_1 = 1, \ldots, x_t\} \subseteq H$ is a transversal of AL in G and $v_i \in X \otimes Z$. Set

$$\phi^*(v) = v_1 .$$

The map $\phi \to \phi^*$ is the familiar reciprocity map. As $\mathbf{k}$-modules, by the map $\phi \to \phi^*$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{k}[H]}(V, (X \otimes Z)|^G) &\simeq \mathrm{Hom}_{\mathbf{k}[H]}(V|_H, (\dim X) Z|^H) \\ &\simeq \mathrm{Hom}_{\mathbf{k}[L]}(V|_L, (\dim X) Z) \\ &\simeq \mathrm{Hom}_{\mathbf{k}[L]}((\dim W) Z, (\dim X) Z) \\ &\simeq \mathrm{Hom}_{\mathbf{k}[L]}(V|_{AL}, X \otimes Z). \end{aligned}$$

So the map $\phi \to \phi^*$ is a $\mathbf{k}$-isomorphism. This map is clearly a map of trivial $\mathbf{k}[L]$-modules. So we need only verify that it is a $\mathbf{k}[A]$-map.

Let $Z_i^* = x_i \otimes_{\mathbf{R}} (X \otimes Z)$. Then $(X \otimes Z)|^G \simeq \sum \oplus Z_i^*$. Let $Z' = Z_1^*$, $Z'' = \sum_{i>1} \oplus Z_i^*$. Notice that $(X \otimes Z)|^G \simeq Z' \oplus Z''$ is an A-decomposition. Let $\pi_1 \colon (X \otimes Z)|^G \to Z'$ be the projection map. If $z_1 \in Z'$, $z_2 \in Z''$ then for $a \in A$ we get

$$a\pi_1 a^{-1}(z_1 + z_2) = a\pi_1(a^{-1}z_1 + a^{-1}z_2) = a(a^{-1}z_1) = z_1 .$$

Now $1 \otimes_{\mathbf{R}} \phi^* = \pi_1 \phi$. So

$$\begin{aligned} a(\pi_1\phi)\, a^{-1} &= (a\pi_1 a^{-1})(a\phi a^{-1}) \\ &= \pi_1(a \cdot \phi). \end{aligned}$$

Thus $a \cdot \phi^* = a\phi^* a^{-1} = (a \cdot \phi)^*$ and we have our $\mathbf{k}[AL]$-isomorphism

$$\mathrm{Hom}_{\mathbf{k}[H]}(V, (X \otimes Z)|^G) \simeq \mathrm{Hom}_{\mathbf{k}[L]}(V|_{AL}, X \otimes Z).$$

But

$$\mathrm{Hom}_{\mathbf{k}[L]}(V\,|_{AL}\,,\, X \otimes Z) \simeq \mathrm{Hom}_{\mathbf{k}[L]}(W \otimes Z, X \otimes Z) \simeq \mathrm{Hom}_{\mathbf{k}}(W, X)$$

are all $\mathbf{k}[A]$-isomorphisms by Theorem 2.1. Putting all of them together we get

$$\mathrm{Hom}_{\mathbf{k}}(W, X) \simeq U$$

is a $\mathbf{k}[A]$-isomorphism.

THEOREM 2.5. *Assume the hypotheses of Theorem 2.4. Then we have the following* $\mathbf{k}[A]$*-isomorphisms*

$$U \simeq \hat{W} \otimes X,$$

where $\hat{W}$ *is the dual of* W*. Further*

$$(J \otimes Z)|^G \simeq V \otimes \hat{W},$$

where J *is the one dimensional trivial* $\mathbf{k}[A]$ *module.*

By [5(51.7)] we may write

$$(J \otimes Z)|^G \simeq V \otimes U_0$$

for some U_0. Applying Theorem 2.3 we obtain

$$U_0 \simeq \mathrm{Hom}_{\mathbf{k}}(W, J) \simeq \hat{W}$$

as $\mathbf{k}[A]$-modules.

Also

$$U \simeq \mathrm{Hom}_{\mathbf{k}}(W, X) \simeq \mathrm{Hom}_{\mathbf{k}}(J, \hat{W} \otimes X) \simeq \hat{W} \otimes X$$

is a $\mathbf{k}[A]$-isomorphism.

THEOREM 2.6. *Assume the hypotheses of Theorem 2.4. If* $(X \otimes Z)\,|^G$ *is irreducible then* $(1 \otimes Z)\,|^G$ *is also irreducible.*

Note that $(X \otimes Z)\,|^G \simeq V \otimes U \simeq V \otimes \hat{W} \otimes X \simeq (1 \otimes Z)\,|^G \otimes X$, where X is an $A \simeq G/H$ module. If $(1 \otimes Z)\,|^G$ reduces then also $(X \otimes Z)\,|^G$ reduces.

3. THE MAIN THEOREM

THEOREM 3.1. *Assume* $\mathbf{k}$ *is an algebraically closed field,* G *a solvable group, and* V *an irreducible* $\mathbf{k}[G]$*-module. The module* V *is primitive if and only if it is quasiprimitive.*

We assume V is primitive first. This case is trivial. Assume $N \triangle G$ and $V|_N = V_1 \dotplus \cdots \dotplus V_t$, where the V_i are homogeneous components. Let S be the stabilizer in G of V_1. Then viewing V_1 as an S module we have

$$V \simeq V_1|^G.$$

This violates the primitivity of V unless $t = 1$ and $S = G$. So $V|_N$ is homogeneous. Therefore V is quasiprimitive.

We now assume that V is quasiprimitive. We assume that 3.1 is false and proceed by induction upon dim V. Among all counterexamples choose one minimizing dim V. That is, G contains a proper subgroup A and V contains a primitive A-module U so that $U|^G \simeq V$. If we factor out the kernel of V we may assume that:

3.2. *V is faithful.*

3.3. *If M is a normal abelian subgroup of G then $M \leqslant Z(G)$.*

Let M be a normal abelian subgroup of G. By quasiprimitivity $V|_M$ is homogeneous. Also $V|_M$ is a multiple of a single faithful absolutely irreducible M module. So M acts upon V as scalar multiplication. Thus $M \leqslant Z(G)$.

3.4. *If P is a normal p subgroup of G such that $AP < G$ then $P \leqslant Z(G)$.*

Assume P is a normal p subgroup of G, $AP < G$, and $P \nleqslant Z(G)$. If M is any characteristic abelian subgroup of P then $M \triangle G$ so that $M \leqslant P \cap Z(G) \leqslant Z(P)$ by 3.3. By [2(2.1), (2.2)] P must be the central product of a cyclic group $Z(P)$ and an extra special group, where $Z(P) \leqslant Z(G)$ by 3.3.

Now $V|_{Z(P)}$ is homogeneous. It is a multiple of a single one dimensional $Z(P)$-module Z. There is a unique irreducible module Z' lying over Z on P. Thus $V|_P$ is a homogeneous multiple of Z'. We let Z^* be a projective extension of Z' to G with factor set α of p-power order ([1, Theorem 2] and an elementary argument). Let $U' = U|^{AP}$. Then $U|_P$ is also a multiple of Z'. So there is a projective module W^* on AP with factor set α^{-1} such that

$$U' \simeq Z^*|_{AP} \otimes W^*.$$

Form the central extension

$$1 \to \langle\alpha\rangle \to G^* \to G \to 1.$$

Then all representations are linear for appropriate subgroups of G^*. If L is a subgroup of G let L^* be its inverse image in G^*. So

$$V \simeq U' |^{G^*} \simeq (Z^* |_{(AP)^*} \otimes W^*) |^{G^*}$$
$$\simeq Z^* \otimes (W^* |^{G^*}).$$

For the moment set $X^* = W^* |^{G^*}$. Now X^* is a projective G-module with factor set α^{-1}.

Now $U' |_P$ is a multiple of $Z^* |_P \simeq Z'$. Thus $W^* |_P$ is linear (i.e., α is trivial) and it is the trivial P-module. Thus $X^* |_P$ is linear and is the trivial P-module since $P \triangle G$. To keep things linear we do a bit of splitting now. Since α^{-1} is the factor set of X^*, X^* is faithful for $\langle\alpha\rangle$ in G^*. Set $P_0^* = P^* \cap \ker X^*$. Note that X^* is trivial for P as a projective module so that

$$P^* = P_0^* \times \langle\alpha\rangle.$$

Thus P_0^* is normal in G^* and is in the kernel of X^*. Let $N^* \geqslant P_0^*$ be a normal subgroup of G^*. Assume

$$X^* |_{N^*} \simeq r(X_1^* \dotplus \cdots \dotplus X_m^*),$$

where the X_i^* are irreducible nonisomorphic components. Thus

$$V |_{N^*} \simeq r(Z^* \otimes X_1^* \dotplus \cdots \dotplus Z^* \otimes X_m^*).$$

Now $Z^* |_{P_0^*}$ is irreducible and X_i^* is trivial on P_0^*. So by Theorem 2.3 $Z^* \otimes X_i^*$ is irreducible. But $V |_{N^*}$ is homogeneous since V is quasiprimitive. So $Z^* \otimes X_i^* \simeq Z^* \otimes X_j^*$ for all i, j. But again by Theorem 2.3 we get $X_i^* \simeq X_j^*$. So $m = 1$ and $X^* |_{N^*}$ is homogeneous. Since P_0^* is in the kernel of X^*, X^* is quasiprimitive on G^*. Now $\dim Z' > 1$ so $\dim X^* < \dim V$. So X^* is primitive. But X^* is induced from W^* on $(AP)^*$. We conclude that $(AP)^* = G^*$ or $AP = G$. This contradicts our assumption that $AP < G$. So 3.4 holds.

If $O_p(G)$ is abelian for every prime p then $F(G) \leqslant Z(G)$ by Theorem 3.3. That is, $G = Z(G)$ and V is obviously primitive. So for some prime $O_p(G)$ is nonabelian. By (3.4) $O_p(G)A = G$, $O_p(G) \nleqslant A$, and $O_p(G) \nleqslant Z(G)$. Note that $O_p(G) \triangle G$.

This argument shows that we may choose P minimal satisfying

(1) $P \triangle G$,
(2) P is a nonabelian p-group,
(3) $P \nleqslant A$; $P \nleqslant Z(G)$.

By 3.4 we have $AP = G$. Also $A \cap P = D(P)$ is the Frattini subgroup $D(P) \triangle G$ and by 3.3 and 3.4, $D(P) \leqslant Z(G)$. Thus $A \cap P = Z(P) \leqslant Z(G) \leqslant A$. Every characteristic abelian subgroup of P is normal in G and by Theorem 3.3 is in $Z(G)$. So P is a central product of $Z(P)$ with an extra special group. So by [2(2.1), (2.2)] we obtain from the minimality of P:

3.5. *P is extra special, $AP = G$, and $A \cap P = Z(P) \leqslant Z(G)$.*

3.6. $C_G(P) = Z(G)$.

Suppose $C_G(P) \nleqslant Z(G)$. Choose Q a normal q-subgroup of G contained in $C_G(P)$ minimal such that $Q \nleqslant Z(G)$. By (3.3) Q is nonabelian. If $q \neq p$ then since $AP = G$ we have $Q \leqslant A$. So $AQ < G$. This contradicts (3.4). So $q = p$ and Q is a p-group. By (3.4) we also have $AQ = G$.

Every characteristic abelian subgroup of Q is in $Z(G)$ by (3.3). So again by the minimality of Q and [2(2.1), (2.2)] Q is extra special. Now $Q \cap P$ is normal in G and centralizes P so $Q \cap P = Z(P)$. Let $R = QP$. Set $\bar{Q} = Q/Z(P)$ and $\bar{P} = P/Z(P)$. Then

$$\bar{R} = R/Z(P) = \bar{Q} \dotplus \bar{P}$$

is a G decomposition into submodules. Both $\bar{Q}$ and $\bar{P}$ are irreducible by minimality.

Let $\bar{T} = (A/Z(P)) \cap \bar{R}$. Since $(A/Z(P)) \cap \bar{P} = 1$ and $(A/Z(P))\bar{P} = G/Z(P)$ we must have

$$\bar{R} = \bar{T} \dotplus \bar{P}$$

another G decomposition into irreducibles where $|\bar{T}| = |\bar{Q}| \neq 1$. Let T be the inverse image of $\bar{T}$ in R. Since R is a p group, T is a p group. Clearly T is normal in G. Also $T \leqslant A$. By 3.4 then $T \leqslant Z(G)$. But $R \cap Z(G) = Z(P)$ so $T \leqslant Z(P)$ or $|T| = 1$. This contradiction proves 3.6.

In what follows we will want to split V into a tensor product of projective G-modules in a special way. We first use the existence of U to split V. Now $U|_{Z(P)}$ is a multiple of a one dimensional module Z. We may extend Z to a projective one dimensional A-module Z^0 with a factor set β of p-power order. This is easily done by letting $x \in A$ act trivially if $x \notin Z(P)$. Note that β is trivial upon $Z(P)$. If we inflate β by letting it be trivial upon P then we obtain a factor set β' of $AP = G$ trivial upon P since $AP/P \simeq A/Z(P)$.

THEOREM 3.7. *$Z^0|^G$ is a projective G-module with factor set β'.*

If $a, b, c, d \in P$; $x, y \in G$ then $\beta'(axb, cyd) = \beta'(x, y)$ since β' is a factor set of G/P. Thus for a transversal $= \{\bar{u}\} \subseteq P$ of A in G and $v \in U$ we have

$$\begin{aligned} x(y(\bar{u} \otimes v)) &= \overline{xyu} \otimes (\overline{xyu}^{-1}x\overline{yu})(\overline{yu}^{-1}y\bar{u})v \\ &= \overline{xyu} \otimes (\overline{xyu}^{-1}xy\bar{u})\, \beta'(x, y)v \\ &= (xy)\, \beta'(x, y)(\bar{u} \otimes v). \end{aligned}$$

We have used the identity

$$\beta'(\overline{xyu}^{-1}\overline{xyu}, \overline{yu}^{-1}\overline{yu}) = \beta'(x, y)$$

remarked upon above. Therefore $Z^0 |^G$ has a projective G-action with factor set β'. This proves 3.7.

We may write

$$U \simeq Z^0 \otimes W^0,$$

where W^0 is a projective A-module with factor set β^{-1}. Since W^0 is trivial upon $Z(P)$, by inflation to G (or $G/P \simeq A/Z(P)$) we obtain a projective G-module W' with factor set $(\beta')^{-1}$. Restriction $W' |_A \simeq W^0$ retrieves W^0 for us. Thus

$$V \simeq U |^G \simeq (W^0 \otimes Z^0) |^G \simeq W' \otimes (Z^0 |^G).$$

3.8. *$V \simeq W' \otimes (Z^0 |^G)$, where $W' |_P$ is trivial and $(Z^0 |^G) |_P$ has trivial factor set.*

Since W^0 is trivial on $Z(P)$, W' is trivial on P. Now β' is trivial on P since β is trivial on $Z(P)$.

Observe that $V |_P \simeq (\dim W')(Z^0 |^G |_P) \simeq (\dim W')[P : Z(P)]^{1/2} Y$, where Y is an irreducible P-module lying over $Z(P)$. Here we have used that fact that Y is uniquely determined by Z up to isomorphism, and the fact that $\dim Y = [P : Z(P)]^{1/2}$ [6(5.5.5)].

We may extend Y to a projective G-module Y' with factor set γ of p-power order [1, Theorem 2]. Further, [3(7.2)] we may assume that if $x \in G$ has p' order then the transformation induced by x upon Y' has determinant equal to one. By [5(51.7)] we may now write:

3.9. *$Z^0 |^G \simeq Y' \otimes X'$, where X' is trivial for P and has factor set $\gamma^{-1}\beta'$.*

3.10. *X', γ and β' are trivial for the p'-elements of $Z(G)$.*

Write $Z(G) = B \times C$ where B is the p'-part of $Z(G)$ and C is a p-group. By our construction of Z^0 (all elements $x \in A/Z(P)$ act trivially upon Z^0) we know β must be trivial upon B. Thus β' is trivial upon B. Since B is trivial upon Z^0, B is trivial upon $Z^0 |^G$. Suppose $x \in B$ and x induces the transformation $\bar{x}$ on Y'. Now $\det \bar{x} = 1$. Further, $\bar{x}$ has degree a power of p. Since $\bar{x}$ is in the center of the action of G on Y', $\bar{x}$ is scalar. Thus $\bar{x}$ is trivial. So B is trivial and linear on Y'. Thus γ is trivial upon B. We conclude that X', γ and β' are all trivial for B.

Choose M so that $M/Z(G)P$ is a G-chief factor. Since $C_G(P) = Z(G)$, $M/Z(G)P$ is a q-section for some prime $q \neq p$. Choose Q a q Sylow subgroup of M in A. Let $N = QP$. Note that $N \triangle G$.

3.11. *γ and β' are trivial upon N.*

To observe that γ is trivial, we remark that $Y'|_N$ is the unique extension of Y to N in which every p' element acts with transformation of determinant 1 [3(7.2)].

Note that $N \triangle G$ so that $N \cap A = QZ(P) \triangle A$. Thus $Q \triangle A$. Now β is trivial upon $QZ(P)$ since $Q \triangle A$ is a $q \neq p$-group. (Recall that $x \in A/Z(P)$ is trivial upon Z^0.) Thus β' is trivial upon $QZ(P) = N$.

The remarkable point of all this is that X', Y', and W' are all linear for N, i.e., they are projective with trivial factor sets.

3.12. *X' is faithful for $Q/Q \cap Z(G)$. $X'|_Q \simeq \hat{Y}'|_Q$, the dual of $Y'|_Q$.*

Now $Z^0|^G|_N \simeq Z^0|_{A\cap N}|^N \simeq Y'|_N \otimes (\hat{Y}'|_Q)$, where $\hat{Y}'$ is the dual of Y' and we view $(\hat{Y}'|_Q)$ as a QP/P-module. This follows from (2.5). Since Y' is trivial for $Q \cap Z(G)$ and ker Y' is in $C_G(P)$ we find that $\hat{Y}'$ is faithful for $Q/Q \cap Z(G)$. But by (2.3) we now have $(\hat{Y}'|_Q) \simeq X'|_N$.

3.13. *The characteristic of* $\mathbf{k}$ *is prime to q.*

Note that $V|_N$ is completely reducible. So also $X'|_N$ is completely reducible. But N acts as $Q/Q \cap Z(G) \simeq N/P(Q \cap Z(G))$ upon X', a q-group. Further Q is nontrivial. Thus (3.13) follows.

We now are in a position where the representation theory of N upon V is ordinary (i.e., "nonmodular"). We consider the module $X' \otimes W'$. Note that $\dim V = [P : Z(P)]^{1/2} \dim X' \otimes W'$ so that $\dim X' \otimes W' < \dim V$. In what follows we will show first, using N, that $X' \otimes W'$ is induced from a proper submodule, and second that $X' \otimes W'$ is quasiprimitive. Since $\dim X' \otimes W' < \dim V$ our induction hypothesis forces a contradiction with Theorem 3.1.

3.14. *$X'|_N$ involves at least two nonisomorphic irreducible direct N summands.*

Suppose $X'|_N$ is homogeneous. Now $X'|_N$ represents N faithfully as $Q/Q \cap Z(G)$. But $Q/Q \cap Z(G) \simeq M/Z(P)P$ is abelian. So Q acts as scalar multiplication upon X'. But $X'|_Q \simeq \hat{Y}'|_Q$ so Q acts as scalar multiplication on Y'. Thus $Q \leqslant C_G(P) \leqslant Z(G)$ an absurdity. So 3.14 holds.

We now form the central extension

$$1 \to \langle\beta'\rangle \times \langle\gamma\rangle \to G^* \to G \to 1.$$

So Y', X', W' are now all with trivial factor set on G^*. Since X', Y', W' are all linear for Z, the inverse image Q^* of Q in G^* in a split extension of $Q_0 \simeq Q$

by $\langle\beta'\rangle \times \langle\gamma\rangle$. Now $Q_0 \ker X' \triangle G^*$. By (3.14) $X' |_{Q_0 \ker X'}$ is not homogeneous. Let X'' be a homogeneous component. Let B^* be the stabilizer in G^* of X''. Then X'' is a B^*-module and

$$X'' |^{G^*} \simeq X'.$$

But then

$$\begin{aligned}(X'' \otimes W' |_{B^*}) |^{G^*} &\simeq (X'' |^{G^*}) \otimes W' \\ &\simeq X' \otimes W'.\end{aligned}$$

We conclude that:

3.15. *$X' \otimes W'$ is induced from a proper subgroup of G^*. That is, $X' \otimes W'$ is not a primitive G^*-module.*

Next we prove that:

3.16. *$X' \otimes W'$ is a quasiprimitive G^*-module.*

Since X'', Y', W' are linear for P, the inverse image P^* of P in G^* splits as

$$P^* \simeq P_0 \times \langle\gamma\rangle \times \langle\beta'\rangle,$$

where $P_0 \simeq P$ and $P_0 \triangle G^*$. Further, P_0 acts as P upon X', Y', W'.

Observe that since V is irreducible, $X' \otimes W'$ is also. Suppose $P_0 \leqslant L^* \triangle G^*$ and

$$(X' \otimes W') |_{L^*} \simeq a(X_1{}^* \oplus \cdots \oplus X_m{}^*),$$

where the $X_i{}^*$ are nonisomorphic irreducible L^*-modules. Now

$$V |_{L^*} \simeq Y' |_{L^*} \otimes a(X_1{}^* \oplus \cdots \oplus X_m{}^*)$$

and $V |_{L^*}$ is homogeneous since V is quasiprimitive. Note that $(X' \otimes W') |_{P_0}$ is trivial and $Y' |_{P_0}$ is irreducible. Now homogeneity of $V |_{L^*}$ forces $Y' |_{L^*} \otimes X_i{}^* \simeq Y' |_{L^*} \otimes X_j{}^*$ since by Theorem 2.3 $Y' |_{L^*} \otimes X_i{}^*$ is irreducible. So by Theorem 2.3 again $X_i{}^* \simeq X_j{}^*$ forcing $m = 1$. But then $(X' \otimes W') |_{L^*}$ is homogeneous. Since $\ker X' \otimes W'$ contains P_0 we conclude that $X' \otimes W'$ is quasiprimitive.

3.17. *Theorem* 3.1 *is valid.*

Since $\dim X' \otimes W' < \dim V$ induction tells us Theorem 3.1 applies to G^* on $X' \otimes W'$. By Theorem 3.16, $X' \otimes W'$ is quasiprimitive so by Theorem 3.1 $X' \otimes W'$ is primitive. This contradicts 3.15 completing Theorem 3.17.

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